Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00034-X

J. Appl. Maths Mechs, Vol. 65, No. 2, pp. 301–309, 2001

0021-8928/01/$—see front matter

WAVE PACKETS IN A THIN CYLINDRICAL SHELL UNDER A NON-UNIFORM AXIAL LOAD†

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(*Received* 18 *April* 2000)

The propagation of localized families of flexural plane waves in a thin elastic shell under an unsteady axial compression, which is non-uniform with respect to the peripheral coordinate, is considered. The shell can be non-circular and open in the peripheral direction. Conditions of hinged support are specified at the edges, which are optionally plane curves. It is assumed that the initial perturbations of the shell (non-zero initial displacements and velocities) are functions which are localized in the neighbourhood of a certain generatrix. The solution is constructed using the complex WKB-method developed in [1] in the form of a superposition of packets of flexural plane waves propagating in the peripheral direction of the shell. This paper differs from [1] in that in addition to taking account of axial stresses, the solution in the direction of the axis of the shell is assumed to be extremely variable. It is shown that it is possible to use the results obtained to investigate the free vibrations of the shell. The unsteady localized vibrations of a cylindrical shell with an elliptic cross-section under axial forces which increase linearly with time are considered as an example.

1. FORMULATION OF THE PROBLEM

Consider a thin cylindrical shell of thickness h. We will introduce an orthogonal system of coordinates x, φ in the middle surface, where x is the axial coordinate and φ is a coordinate in the directrix, introduced in such a manner that the first quadratic form is $d\sigma^2 = R^2(dx^2 + d\varphi^2)$, where R is the characteristic dimension of the middle surface of the shell. Then, $R_2 = R/k(\varphi)$ is the radius of curvature. The shell can be open in a peripheral direction and optionally has plane edges. The middle surface occupies the domain $x_1(\varphi) \leqslant x \leqslant x_2(\varphi)$ when $\varphi_1 \leqslant \varphi \leqslant \varphi_2$.

We assume that the shell is loaded along the peripheral coordinate with axial stresses $T_1^\circ(\varphi, t)$ which vary slowly with time, where the compressive stress does not reach its critical value at which the shell loses stability [2, 3].

Assuming that there is considerable variability in the stress-strain state of the shell in the direction of the x and φ coordinates, we will use the system of equations from the theory of hollow shells which takes account of the existence of dynamic stresses due to axial forces [4],

$$\mu^2\Delta^2 W + T_1(\varphi, t)\frac{\partial^2 W}{\partial x^2} - k(\varphi)\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 W}{\partial t^2} = 0$$
$$\mu^2\Delta^2 F + k(\varphi)\frac{\partial^2 W}{\partial x^2} = 0 \tag{1.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varphi^2}, \quad \mu^4 = \frac{h^2}{12(1-\nu^2)R^2}, \quad W = \mu^2\frac{W_*}{R}, \quad F = \frac{\mu^{-2}F_*}{hE}$$
$$T_1^\circ = -Eh\mu^2 T_1, \quad t = t_*/t_c, \quad t_c^2 = \frac{R^2\rho}{Eh\mu^2}$$

Here W_*, F_* are the normal deflection and stress function, t_* is the time, $0 < \mu$ is a small parameter, h is the shell thickness, E, ν and ρ are Young's modulus, Poisson's ratio and the density of the shell material respectively, T_1° is the initial longitudinal stress and t_c is the characteristic time.

†*Prikl. Mat. Mekh.* Vol. 65, No. 2, pp. 308–316, 2001.

We shall assume that the functions $x_1(\varphi), x_2(\varphi), T_1(\varphi, t), k(\varphi)$ are differentiable a sufficient number of times with respect to φ and t and, together with their derivatives, are of the order of unity when $\mu \to 0$.

At the edges of the shell $x = x_1(\varphi), x = x_2(\varphi)$, we consider the Navier equations [2], corresponding to a hinged support,

$$W = \frac{\partial^2 W}{\partial x^2} = 0, \quad F = \frac{\partial^2 F}{\partial x^2} = 0 \tag{1.2}$$

We consider the initial conditions

$$W|_{t=0} = W_0^*(x, \varphi, \mu)\Phi_0, \quad \dot{W}|_{t=0} = i\mu^{-1}V_0^*(x, \varphi, \mu)\Phi_0 \tag{1.3}$$

$$\Phi_0 = \Phi_0(\varphi, \mu) = \exp\left\{i\mu^{-1}\left(a_0\varphi + \frac{1}{2}b_0\varphi^2\right)\right\}, \quad \mathrm{Im}\, b_0 > 0$$

where a_0 is a real number and W_0^*, V_0^* are complex functions such that

$$\frac{\partial^m W_0^*}{\partial x^m}, \quad \frac{\partial^m V_0^*}{\partial x^m} \sim \mu^{-m} \text{ when } \mu \to 0, \quad m = 0, 1, \dots \tag{1.4}$$

which have a finite number of oscillations in the direction of φ with a variability $\mu^{-1/2}$ and satisfy boundary conditions (1.2). The real and imaginary parts of the functions in conditions (1.3) specify a pair of initial wave packets with a variability of the order of μ^{-1} in the direction of the x and φ coordinates, which are concentrated in the neighbourhood of the generatrix $\varphi = 0$. The nature of the occurrence of wave packets of the type (1.3) has been studied previously in [5, 6] in an investigation of the parametric instability of a cylindrical shell with variable parameters under a periodic axial force.

The aim of this paper is to investigate the reaction of a cylindrical shell to initial perturbations of the type of (1.3) in the case of axial forces T_1 which vary slowly with time and along the peripheral coordinate.

2. METHOD OF SOLUTION

Taking into account the large variability of the stress-strain state in the direction of the longitudinal coordinate, we carry out a scale expansion by putting

$$s = \mu^{-1}x \tag{2.1}$$

In order to solve initial-boundary-value problem (1.1)–(1.3) we will use the method developed in [1] for the case of a small number of waves in the direction of the shell axis.

Consider the sequence of functions

$$z_n(s, \varphi) = \sin\{\lambda_n(\varphi)[s - s_1(\varphi)]\}, \quad \lambda_n = \frac{\pi n}{s_2(\varphi) - s_1(\varphi)}, \quad n = 1, 2, \dots \tag{2.2}$$

Here and henceforth $s_i(\varphi) = \mu^{-1}x_i(\varphi)$, $i = 1, 2$.

The functions $\tilde{W}_0^*$, $\tilde{V}_0^*$ where $\tilde{W}_0^*(s, \varphi, \mu) = W_0^*(x, \varphi, \mu)$, $\tilde{V}_0^*(s, \varphi, \mu) = V_0^*(x, \varphi, \mu)$ for any $\varphi \in [\varphi_1, \varphi_2]$, can be expanded with respect to the system of functions $\{z_n\}$ in series which are uniformly convergent in the interval $[s_1(\varphi), s_2(\varphi)]$ [7]

$$\tilde{W}_0^* = \sum_{n=1}^{\infty} W_{n0}z_n, \quad W_{n0} = \int_{s_1(\varphi)}^{s_2(\varphi)} \tilde{W}_0^* z_n ds$$

$$\tilde{V}_0^* = \sum_{n=1}^{\infty} V_{n0}z_n, \quad V_{n0} = \int_{s_1(\varphi)}^{s_2(\varphi)} \tilde{V}_0^* z_n ds \tag{2.3}$$

Suppose that W_{n0}, V_{n0} are polynomials of the argument $\mu^{-1/2}\varphi$ with coefficients which depend regularly on $\mu^{-1/2}$. Then, W_{n0}, V_{n0} can be represented in the form of the series

$$W_{n0}=\sum_{m=0}^{\infty}\mu^{m/2}w_{nm}^{\circ}(\zeta),\quad V_{n0}=\sum_{m=0}^{\infty}\mu^{m/2}v_{nm}^{\circ}(\zeta);\quad \zeta=\mu^{-1/2}\varphi \tag{2.4}$$

where w_{hm}°, v_{nm}° are polynomials of degrees M_{nm} with (in general) complex coefficients.

Following the approach described previously in [1], we will seek a solution in the form of a superposition of wave packets propagating in the peripheral direction

$$W=\sum_{n=1}^{N}W_n,\quad F=\sum_{n=1}^{N}F_n \tag{2.5}$$

where W_n, F_n are the required functions, which are localized in a certain time interval $t\in[0,T]$ in the neighbourhood of the generatrix $\varphi=q_n(t)$. Here, $q_n(t)$ is the centre of the nth wave packet, which corresponds to the pair of functions W_n, F_n. The required function $q_n(t)$ is assumed to be doubly differentiable such that $q_n(0)=0$.

Each of the n wave packets is subject to the initial conditions

$$W_n\,|_{t=0}=\sum_{m=0}^{\infty}\mu^{m/2}w_{nm}^{\circ}(\zeta)z_n\Phi_0,\quad \dot W_n\,|_{t=0}=i\mu^{-1}\sum_{m=0}^{\infty}\mu^{m/2}v_{nm}^{\circ}(\zeta)z_n\Phi_0 \tag{2.6}$$

We will now change to a moving system of coordinates connected with the centre of the nth wave packet

$$\varphi=q_n(t)+\mu^{1/2}\xi_n \tag{2.7}$$

Taking account of substitutions (2.1) and (2.7), we write system of equations (1.1) for the nth wave packet in the form

$$\begin{gathered}\frac{\partial^4W_n}{\partial s^4}+2\mu\frac{\partial^4W_n}{\partial s^2\partial\xi_n^2}+\mu^2\frac{\partial^4W_n}{\partial\xi_n^4}+T_1\frac{\partial^2W_n}{\partial s^2}-k\frac{\partial^2F_n}{\partial s^2}+\mu^2\frac{\partial^2W_n}{\partial t^2}-\\ -2\mu^{3/2}\dot q_n\frac{\partial^2W_n}{\partial t\partial\xi_n}+\mu\dot q_n^2\frac{\partial^2W_n}{\partial\xi_n^2}-\mu^{3/2}\ddot q_n\frac{\partial W_n}{\partial\xi_n}=0\\ \frac{\partial^4F_n}{\partial s^4}+2\mu\frac{\partial^4F_n}{\partial s^2\partial\xi_n^2}+\mu^2\frac{\partial^4F_n}{\partial\xi_n^4}+k\frac{\partial^2W_n}{\partial s^2}=0\end{gathered} \tag{2.8}$$

Here and henceforth, a dot denotes differentiation with respect to time.

The boundary conditions on the boundaries $s=s_1(\varphi)$, $s=s_2(\varphi)$ retain the form (1.2).

We shall seek a solution of problem (1.2), (2.6), (2.8) in the form of the expansion [1]

$$W_n=\sum_{m=0}^{\infty}\mu^{m/2}w_{nm}(s,\xi_n,t)\Phi_n,\quad F_n=\sum_{m=0}^{\infty}\mu^{m/2}f_{nm}(s,\xi_n,t)\Phi_n \tag{2.9}$$

$$\Phi_n=\exp\left\{i\left[\mu^{-1}\int_0^t\omega_n(\tau)d\tau+\mu^{-1/2}p_n(t)\xi_n+\frac{1}{2}b_n(t)\xi_n^2\right]\right\}$$

Here, $\omega_n(t)$, $p_n(t)$, $b_n(t)$ are doubly differentiable functions of the order of unity, $\operatorname{Im}\,b_n(t)>0$ for any $t\in[0,T]$, and W_{nm}, f_{nm} are polynomials in ξ_n. The mechanical meaning of the parameters ω_n, p_n, b_n has been discussed previously [1].

We expand the coefficients of system (2.8) in Taylor series in powers of $\mu^{1/2}\xi$ in the neighbourhood of the centre of the nth wave packet $\varphi=q_n(t)$. On substituting formulation (2.9) into (2.8) and equating the coefficients of like powers of $\mu^{1/2}$ to zero, we arrive at the sequence of equations

$$\sum_{j=0}^{m}\mathbf{L}_{nj}\mathbf{X}_{nm-j}=0,\quad m=0,1,2,\ldots \tag{2.10}$$

where

$$\mathbf{L}_{n0}=\begin{Vmatrix} l_{n11} & l_{n12}\\ l_{n21} & l_{n22}\end{Vmatrix},\quad \mathbf{X}_{nk}=(w_{nk},f_{nk})^T$$

$$l_{n11}=\left(\frac{\partial^2}{\partial s^2}-p_n^2\right)^2+T_1\frac{\partial^2}{\partial s^2}-(\omega_n-\dot q_n p_n)^2$$

$$l_{n12}=-l_{n21}=-k\frac{\partial^2}{\partial s^2},\quad l_{n22}=\left(\frac{\partial^2}{\partial s^2}-p_n^2\right)^2$$

$$\mathbf{L}_{n1}=b_n\xi_n\mathbf{L}_p+\xi_n\mathbf{L}_q+\dot p_n\xi_n\mathbf{L}_\omega-i\mathbf{L}_p\frac{\partial}{\partial\xi_n}$$

$$\mathbf{L}_{n2}=\frac{b_n^2\xi_n^2}{2}\mathbf{L}_{pp}+b_n\xi_n^2\mathbf{L}_{pq}+\frac{\xi_n^2}{2}\mathbf{L}_{qq}+\frac{\dot p_n^2\xi_n^2}{2}\mathbf{L}_{\omega\omega}+\dot p_n\xi_n^2\mathbf{L}_{\omega q}+\dot p_n b_n\xi_n^2\mathbf{L}_{\omega p}+$$

$$+\frac{\dot b_n\xi_n^2}{2}\mathbf{L}_\omega-\frac{1}{2}\mathbf{L}_{pp}\frac{\partial^2}{\partial\xi_n^2}-ib_n\xi_n\mathbf{L}_{pp}\frac{\partial}{\partial\xi_n}-i\xi_n\mathbf{L}_{pq}\frac{\partial}{\partial\xi_n}-i\dot p_n\xi_n\mathbf{L}_{\omega p}\frac{\partial}{\partial\xi_n}-$$

$$-i\mathbf{L}_\omega\frac{\partial}{\partial t}-\frac{ib_n}{2}\mathbf{L}_{pp}-\frac{i\dot\omega_n}{2}\mathbf{L}_{\omega\omega}-i\dot p_n\mathbf{L}_{\omega p}+\mathbf{S}_n,\quad \mathbf{S}_n=\begin{Vmatrix}-i\ddot q_n p_n & 0\\ 0 & 0\end{Vmatrix}$$

Here and below the subscripts p, q and ω denote differentiation with respect to p_n, q_n, ω_n and the superscript T denotes transposition. The functions k and T_1 and their derivatives are taken when $\varphi = q_n(t)$.

Substituting series (2.9) into condition (1.2) and expanding the functions $s_1(\varphi)$, $s_2(\varphi)$ in Taylor series in powers of $\mu^{1/2}\xi_n$ we obtain the boundary conditions for the vector functions $\mathbf{X}_{nm}$ when $s = s_i(q_n(t))$ $(i = 1, 2)$

$$\mathbf{X}_{n0}=0,\quad \frac{\partial^2\mathbf{X}_{n0}}{\partial s^2}=0 \tag{2.11}$$

$$\mathbf{X}_{n1}+\xi_n s_i'\frac{\partial\mathbf{X}_{n0}}{\partial s}=0,\quad \frac{\partial^2\mathbf{X}_{n1}}{\partial s^2}+\xi_n s_i'\frac{\partial^3\mathbf{X}_{n0}}{\partial s^3}=0 \tag{21.2}$$

$$\mathbf{X}_{n2}+\xi_n s_i'\frac{\partial\mathbf{X}_{n1}}{\partial s}+\frac{1}{2}\xi_n^2\left(s_i''\frac{\partial\mathbf{X}_{n0}}{\partial s}+s_i'\frac{\partial^2\mathbf{X}_{n0}}{\partial s^2}\right)=0$$

$$\frac{\partial^2\mathbf{X}_{n2}}{\partial s^2}+\xi_n s_i'\frac{\partial^3\mathbf{X}_{n1}}{\partial s^3}+\frac{1}{2}\xi_n^2\left(s_i''\frac{\partial^3\mathbf{X}_{n0}}{\partial s^3}+s_i'\frac{\partial^4\mathbf{X}_{n0}}{\partial s^4}\right)=0 \tag{2.13}$$

Hence, initial-boundary-value problem (1.2), (2.6), (2.8), which is two-dimensional with respect to the coordinates has been reduced to a sequence of one-dimensional boundary-value problems, which are considered in a moving generatrix $\varphi = q_n(t)$.

3. SOLUTION OF THE BOUNDARY-VALUE PROBLEMS

We shall seek the solutions of the sequence of boundary-value problems (2.10)–(2.13) in the form

$$\mathbf{X}_{nm}=P_{nm}(\xi_n,t)\mathbf{Y}_n(q_n(t))z_n(s,q_n(t))+\mathbf{X}_{nm}^{(p)}(s,\xi_n,t),\quad m=0,1,2,\ldots \tag{3.1}$$

where P_{nm} is a polynomial in ξ_n, $\mathbf{Y}_n$ is an unknown column vector and $\mathbf{X}_{nm}^{(p)} = (w_{nm}^{(p)}, f_{nm}^{(p)})^T$ is a certain particular solution of the inhomogeneous boundary-value problem which corresponds to the number m. Note that $\mathbf{X}_{n0}^{(p)} = (0, 0)^T$.

Considering the zeroth approximation ($m = 0$), we find

$$\omega_n=\dot q_n p_n-H_n(t,p_n,q_n) \tag{3.2}$$

$$H_n = \pm\sqrt{(\lambda_n^2 + p_n^2)^2 - T_1\lambda_n^2 + \Lambda_n^2}, \quad \mathbf{Y}_n = (1, \Lambda_n)^T; \quad \Lambda_n = k\lambda_n^2(\lambda_n^2 + p_n^2)^{-1} \tag{3.3}$$

where H_n is the Hamiltonian corresponding to the nth wave packet. All the functions in relations (3.3) are calculated for $\varphi = q_n(t)$.

When $m = 1$, we have an inhomogeneous boundary-value problem in $\mathbf{X}_{n1}$. The conditions for a solution of this problem to exist lead to a Hamiltonian system in p_n, q_n [1]

$$\dot{q}_n = H_p(t, p_n, q_n), \quad \dot{p}_n = -H_q(t, p_n, q_n) \tag{3.4}$$

with the initial conditions

$$q_n(0) = 0, \quad p_n(0) = a_0$$

In this approximation, we find

$$\mathbf{X}_{n1}^{(p)} = -i\frac{\partial(\mathbf{Y}_n z_n)}{\partial p_n}\frac{\partial P_{n0}}{\partial \xi_n} + \xi_n\left(b_n\frac{\partial(\mathbf{Y}_n z_n)}{\partial p_n} + \frac{\partial(\mathbf{Y}_n z_n)}{\partial q_n}\right)P_{n0} \tag{3.5}$$

and the polynomials P_{n0}, P_{n1} remain unknown.

On considering the conditions for the existence of a solution of the boundary-value problem in $\mathbf{X}_{n2}$ which arises in the second approximation ($m = 2$), we obtain the Riccati equation [1]

$$\dot{b}_n + H_{pp}b_n^2 + 2H_{pq}b_n + H_{qq} = 0 \tag{3.6}$$

where $b_n(0) = b_0$, and the amplitude equation for determining P_{n0} is

$$h_{n0}\frac{\partial^2 P_{n0}}{\partial \xi_n^2} + h_{n1}\xi_n\frac{\partial P_{n0}}{\partial \xi_n} + h_{n2}\frac{\partial P_{n0}}{\partial t} + h_{n3}P_{n0} = 0 \tag{3.7}$$

$$h_{n0}(t) = H_{pp}, \quad h_{n1}(t) = 2i(b_n H_{pp} + H_{pq}), \quad h_{n2} = 2i$$

$$h_{n3}(t) = iH_n^{-1}\left[b_n H_n H_{pp} - 2H_p H_q - \dot{\omega}_n + \ddot{q}_n p_n + \frac{l'(q_n)}{l(q_n)}H_n H_p + \right.$$

$$\left. + \frac{2}{l(q_n)}\int_{s_1(q_n)}^{s_2(q_n)}\left(\frac{\partial l_{n11}}{\partial p_n} - \Lambda_n^2\frac{\partial l_{n22}}{\partial p_n}\right)z_q z_n ds\right], \quad l(q_n) = s_2(q_n) - s_1(q_n)$$

Here and henceforth, a prime denotes differentiation with respect to φ.

4. SOLUTION OF THE AMPLITUDE EQUATION

The solution of amplitude equation (3.7) in the form of the polynomial

$$P_{n0} = \sum_{k=0}^{M_{n0}} A_{nk}(t)\xi_n^k$$

has been presented previously [1]. Here, we consider another representation of the solution in terms of Hermite polynomials which, in a special case, will be convenient for describing the free vibrations of a prestressed shell close to a "weak" generatrix.

We make the change of variables $y_n = \Theta_n(t)\xi_n$, where

$$\Theta_n(t) = \frac{\exp[-\int(h_{n1}/h_{n2})dt]}{\sqrt{4\int(h_{n0}/h_{n2})\exp[-2\int(h_{n1}/h_{n2})dt]dt}} \tag{4.1}$$

As a result, Eq. (3.7) takes the form

$$\frac{\partial^2 P_{n0}}{\partial y_n^2} - 2y_n \frac{\partial P_{n0}}{\partial y_n} + \frac{h_{n2}}{h_{n0}\Theta_n^2}\frac{\partial P_{n0}}{\partial t} + \frac{h_{n3}}{h_{n0}\Theta_n^2} P_{n0} = 0 \tag{4.2}$$

Applying the method of separation of variables to (4.2) and taking account of the linearity of Eq. (3.7), we have

$$P_{n0}(\xi_n, t; c_{nm}) = \sum_{m=0}^{M_{n0}} c_{nm}\chi_{nm}(t)\mathbf{H}_{nm}[\Theta_n(t)\xi_n] \tag{4.3}$$

$$\chi_{nm}(t) = \frac{\{4\int(h_{n0}/h_{n2})\exp[-2\int(h_{n1}/h_{n2})dt]dt\}^{m/2}}{\exp[\int(h_{n3}/h_{n2})dt]}$$

where $\mathbf{H}_{nm}$ is a Hermite polynomial of degree m and c_{nm} are arbitrary constants, which are determined from the initial conditions.

5. ANALYSIS OF THE SOLUTION

The above solution is the asymptotically principal approximation of the solution of the problem, apart from quantities $O(\mu^{1/2})$ and is the superposition of N wave packets propagating in the peripheral direction. At the initial instant of time, each nth wave packet decomposes into an n^+th and an n^-th wave packet which, when $t = 0$, hae velocities $v_g = \dot{q}_n^{\pm}(t)$ that are equal in modulus and opposite in sign. The width of the wave packet is a quantity of the order of $\mu^{1/2}/\mathrm{Im}\,b_n^{\pm}(t)$. The signs $\pm$ correspond to the sign in the first expression of (3.3).

The unknown constants $c_{nm}^{\pm}$ in (4.3) are determined using the formula

$$c_{nm}^{\pm} = \frac{1}{2^{m+1}m!\sqrt{\pi}\chi_{nm}(0)}\int_{-\infty}^{+\infty} e^{-\varsigma^2}\mathbf{H}_{nm}[\Theta_n(0)\varsigma]\left[w_{n0}^{\circ}(\varsigma) \mp \frac{v_{n0}^{\circ}(\varsigma)}{\omega_n(0)}\right]d\varsigma \tag{5.1}$$

The solution obtained here generalizes the solution obtained previously in [1], since the Hamiltonian (3.3) does not have a singularity when $p_n = 0$ and the constraint $p_n(t) \neq 0$; which is required in [1], drops out.

A steady wave packet. The proposed method contains the possibility of finding the characteristic modes of vibration of a shell, concentrated in the neighbourhood of a "weak" generatrix [8], in the case of a steady axial stress. Suppose that a cylindrical shell of variable curvature with straight edges $s_1(\varphi), s_2(\varphi) =$ const is under a steady axial stress $T_1(\varphi)$, which is non-uniform with respect to the peripheral coordinate.

Consider the system

$$H_p = 0, \quad H_q = 0 \tag{5.2}$$

and the equation

$$H_{pp}b^2 + 2H_{pq}b + H_{qq} = 0 \tag{5.3}$$

which are the degenerate steady analogues of the Hamiltonian system and the Riccati equation, respectively. System (5.2) has two solutions

$$p = p_w = 0, \quad 2k(\varphi_w)k'(\varphi_w) - \lambda_n^2 T_1'(\varphi_w) = 0 \tag{5.4}$$

and

$$p = p_w = \sqrt{k^{1/2}(\varphi_w)\lambda_n - \lambda_n^2}, \quad 2k'(\varphi_w) - T_1'(\varphi_w) = 0 \tag{5.5}$$

The last two equations in (5.4) and (5.5) serve to determine the "weak" generatrix $\varphi = \varphi_w$. Here, Eqs (5.4) have to be used when

$$\lambda_n > k^{1/2}(\varphi_w), \quad 2k'^2(\varphi_w) + 2k(\varphi_w)k''(\varphi_w) - T_1''(\varphi_w)\lambda_n^2 > 0 \tag{5.6}$$

and Eqs (5.5) when

$$\lambda_n < k^{1/2}(\varphi_w), \quad 4k''(\varphi_w) - 2T_1''(\varphi_w) > 0 \tag{5.7}$$

Solving equation (5.3), we find

$$b_w = \frac{i\lambda_n}{2}\sqrt{\frac{2k'^2(\varphi_w) + 2k(\varphi_w)k''(\varphi_w) - T_1''(\varphi_w)\lambda_n^2}{\lambda_n^4 - k^2(\varphi_w)}}$$

$$b_w = \frac{\lambda_n}{8}\frac{T_1'(\varphi_w)k^{-1/2}(\varphi_w) + i\sqrt{4k''(\varphi_w) - 2T_1''(\varphi_w)}}{\sqrt{k^{1/2}(\varphi_w)\lambda_n - \lambda_n^2}} \tag{5.8}$$

for cases (5.6) and (5.7), respectively. Without loss of generality, we next assume that $\varphi_w = 0$. Now, suppose that $p_n(0) = p_w, b_n(0) = b_w$. Then, the solution of Hamiltonian system (3.4) and Riccati equation (3.6) remain constant:

$$p_n(t) = p_w, \quad q_n(t) = 0, \quad b_n(t) = b_w, \quad \forall t \in [0, T].$$

We will determine the polynomial P_{n0} in accordance with relation (4.3). As a result, we obtain

$$(W_n, F_n)^T = \mathbf{Y}_n z_n \exp\left\{i\mu^{-1}\left(\omega_w t + p_w\varphi + \frac{1}{2}b_w\varphi^2\right)\right\} \times$$

$$\times\left\{\sum_{m=0}^{M_{n0}}(c_{nm}^+ + c_{nm}^-)\left(\frac{iH_{pp}}{b_w H_{pp} + H_{pq}}\right)^{m/2}\mathbf{H}_{nm}\left[\mu^{-1/2}\sqrt{\frac{b_w H_{pp} + H_{pq}}{H_{pp}}}\varphi\right] + O(\mu^{1/2})\right\} \tag{5.9}$$

$$\omega_w = H_n(0, p_w, 0) - \mu\left(\left(m + \frac{1}{2}\right)b_w H_{pp} + mH_{pq}\right)$$

In (5.9), $c_{nm}^{\pm}$ are found using formula (5.1), all the functions are calculated when $p_n = p_w$, $q_n = 0$ and M_{n0} is the degree of the polynomials $w_{n0}^\circ(\zeta)$, $v_{n0}^\circ(\zeta)$ from the initial conditions.

Function (5.9) is the superposition of $M_{n0} + 1$ characteristic modes of vibration of a cylindrical shell which has been prestressed in the axial direction [5]. The solution is a steady wave packet with its centre at the "weak" generatrix $\varphi_w = 0$. Analogous solutions, describing the free vibrations of a shell close to a "weak" generatrix, have been constructed in [8] in the case of a small number of waves along the generatrix. Formula (5.9) holds if $\lambda_n \neq k^{1/2}(0)$. The case of free vibrations when $\lambda_n \sim k^{1/2}(0)$ has been considered previously [6].

An unsteady wave packet. We will now analyse the solution in the case of a thin circular cylindrical shell with straight edges $s_1(\varphi), s_2(\varphi) = \text{const}$ loaded with a steady axial stress $T_1(\varphi)$ which is non-uniform along the circumference. Only some of the deductions made here will be presented below. Analysis of the Hamiltonian system shows that, if

$$k^2\lambda_n^4 > (a_0^2 + \lambda_n^2)^4, \quad T_1'(\varphi) < 0 \text{ when } 0 < \varphi < \varphi_2, \quad T_1(\varphi) = T_1(-\varphi) \tag{5.10}$$

$$\inf_{[0,\varphi_2]} T_1(\varphi) < A, \quad A = 2k - [H_n(0, a_0, 0)]^2\lambda_n^{-2} \tag{5.11}$$

then a t_r exists such that $v_g(0)v_g(t) > 0$ when $0 < t < t_r$, $v_g(t_r) = 0$, where $T_1[q_n(t_r)] = A$; $v_g(0)v_g(t) < 0$ when $t_r < t < 3t_r$, $v_g(3t_r) = 0$ and $T_1[q_n(3t_r)] = A$; $q_n(3t_r) = -q_n(t_r)$; $v_g(0)v_g(t) > 0$ when $3t_r < t < 4tr$ and $v_g(4t_r) = v_g(0)$. Here, p_n, q_n, ω_n are periodic functions of time with period $4t_r$.

The periodicity of the coefficients of the Riccati equation (3.6) follows from this and the question arises as to the nature of the solutions of both the Riccati equation and of the amplitude equation (3/7), when conditions (5.10) and (5.11) are satisfied.

6. EXAMPLES

1. Consider a flexibly supported circular cylindrical shell with straight edges subjected to a steady, non-uniform axial load. Suppose that

$$T_1 = 1 + \cos 2\varphi, \quad s_1 = 0, \quad s_2 = 4, \quad k = 2, \quad a_0 = {}^2/_3, \quad b_0 = i$$

$$\tilde{W}_0^* = w_{n0}^\circ z_n, \quad \tilde{V}_0^* = v_{n0}^\circ z_n$$

In this case, the initial wave packet is located at the "weakest" generatrix, where the compression T_1 is a maximum Calculations were carried out for

$$h = 0{,}02, \quad R = 50, \quad \nu = 0{,}3, \quad w_{n0}^\circ = 1, \quad v_{n0}^\circ = 1, \quad n = 1 \tag{6.1}$$

Conditions (5.10) and (5.11) are satisfied for these parameter values.

The graphs of the functions p_1^+ (t), q_1^+ (t), ω_1^+ (t) (curves *1–3* respectively) in Fig. 1 show that the behaviour of the 1^+-st wave packet agrees completely with the results of the analysis: multiple reflection of the wave packet is observed and the functions p_1^+, q_1^+, ω_1^+ are periodic. Graphs of the functions Im $b_1^+(t)$, Re $w_{10\max}^+$ (curves *4* and *5*) show that each reflection is accompanied by focussing of the wave packet (by in increase in Im b_1^+) and an increase in the amplitude of the vibrations. An approximated calculation of the monodromy matrix [9] for a linear system of equations, which is equivalent to (3.6), enabled us to establish that, in this case, the function $b_1^\mp(t)$ is not periodic and the interval between two even-numbered maxima of the function Im $b_1^+(t)$ only slightly exceeds the period $4t_r$ of the functions p_1^+, q_1^+, ω_1^+.

2. Consider a flexibly supported cylindrical shell with straight edges which has an ellipse with semi-axes e_1, e_2 ($e_1 < e_2$) as the directrix. Suppose the initial wavepacket is located in the generatrix corresponding to the smallest curvature (the "weakest" generatrix). Note that there will be two such generatrices here. Suppose the shell is loaded with an axial stress $T_1 = t$ which slowly varies linearly with time. Calculations were carried out in the interval $t \in [0, T_{cr})$ where T_{cr} is the critical value of the axial stress [2], which corresponds to loss of stability of the shell. As the initial wave packet, we consider one of the characteristic modes of the low-frequency vibrations of a shell which is stress free ($T_1 = 0$). We put

$$e_1 = 1, \quad e_2 = 1{,}5, \quad s_1 = 0, \quad s_2 = 5$$

$$a_0 = \sqrt{k^{1/2}(0)\lambda_n - \lambda_n^2}, \quad b_0 = \frac{i\lambda_n}{4}\frac{\sqrt{k''(0)}}{\sqrt{k^{1/2}(0)\lambda_n - \lambda_n^2}}$$

$$\tilde{W}_0^* = w_{n0}^\circ z_n, \quad \tilde{V}_0^* = v_{n0}^\circ z_n$$

The values of the remaining parameters are the same as in (6.1).

The same notation is used in Fig. 2 as in Fig. 1. We see that the functions p_1^+, q_1^+, Im b_1^+ are constant, that is, an increasing axial stress T_1 has no effect on the vibration modes but leads to a reduction in the frequency of the vibrations $|\omega_1^+|$ and to an unlimited increase in their amplitudes. This indicates the possibility of a dynamic loss of stability at values of $T_1(t)$ less than the critical value T_{cr} and the need for a further investigation in a non-linear formulation.

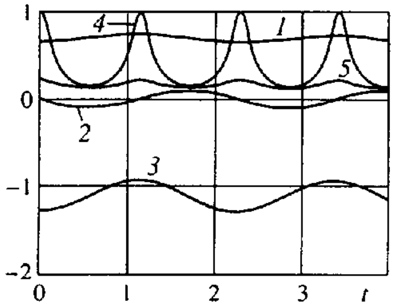

Fig. 1

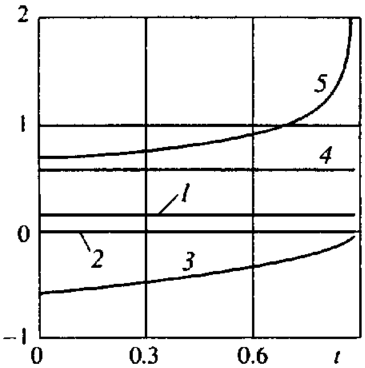

Fig. 2

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Translated by E.L.S